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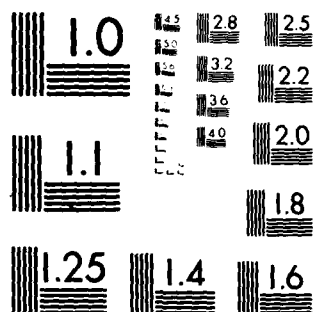
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## Multivariate Harmonic New Better Than Used In Expectation Distributions

A.P. Basu and Nader Ebrahimi  
University of Missouri-Columbia

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ABSTRACT

Various definitions of multivariate harmonic new better than used in expectation (MHNBE) life distributions are introduced and their interrelationship is studied. These are multivariate generalizations of the largest available univariate class of distributions with aging properties. Examples are given to illustrate these concepts. Various closure properties of MHNBE distributions are proved.

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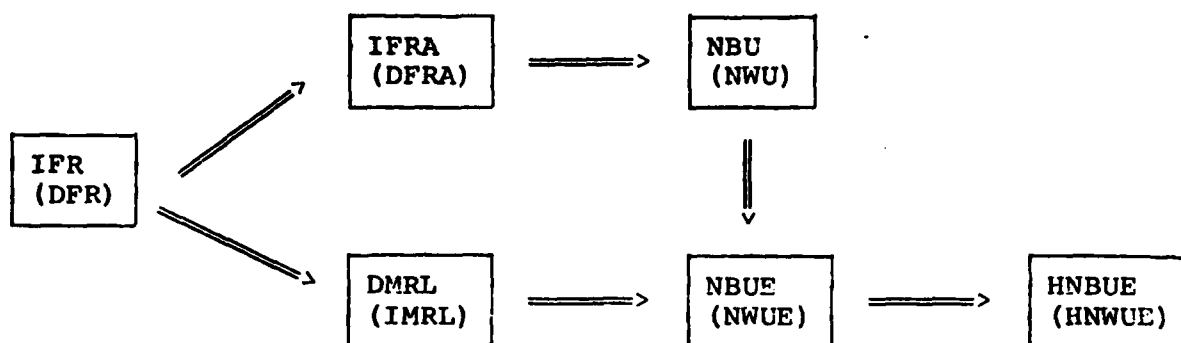
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1. Introduction. In reliability theory various concepts of (univariate) aging or wearout have been proposed to study lifetimes of systems and components. The five most commonly studied class of distributions are the following:

1) The increasing failure rate class (IFR); 2) the increasing failure rate average class (IFRA); 3) the new better than used class (NBU); 4) decreasing mean residual class (DMRL); and 5) the new better than used in expectation (NBUE) class. For a description of some of these classes see Barlow and Proschan (1975). Recently Rolski (1975) proposed a new class of distributions called the harmonic new better than used in expectation (HNBUE) class which will be defined later. Each of the above six classes have their dual with standard nomenclature. The dual of HNBUE class is said to be harmonic new worse than used in expectation (HNWUE). Klefsjö (1980) has studied the properties of HNBUE (HNWUE) classes of distributions. He has proven several closure theorems for this class and the following chain of implication exists among the six classes of distributions.

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Thus HNBUE is the largest available class of distributions with aging property.

Recently attention has been directed towards extending the concepts of univariate aging to the multivariate case, (see for example Harris (1970), Basu (1971), and Brindly and Thompson (1972)), and using those multivariate concepts to define corresponding multivariate class of distributions. Buchanan and Singpurwalla (1977), Block and Savits (1980), and Ghosh and Ebrahimi (1980) have considered the cases for multivariate IFR, IFRA, NBU, DMRL, and NBUE distributions. The purpose of this note is to propose multivariate versions of HNBUE distributions (MHNBU). It is shown that these are the largest available classes, and include the class of MHNBU proposed by Klefsjö (1980) for the bivariate case.

In section 2 of this paper, we have introduced the various definitions of the MHNBU involving a certain hierarchy, and have described their physical implications. Our definitions of the MHNBU are different from Klefsjö's definitions of MHNBU. We have compared our definitions of MHNBU with Klefsjö definitions of MHNBU. We have also examined in this section how several important classes of life distributions satisfy one or the other definition of MHNBU.



The dual of the HNBUE is HNWUE. In section 2, multivariate HNWUE (MHNWUE) definitions are also given parallel to those for the MHNBUE.

Various closure properties of MHNWUE distributions under different definitions are studied in section 3. It is known that in the univariate case NBUE class is included in the HNBUE class. It is examined in section 4 how far the MHNWUE distributions as introduced by Buchanan and Singpurwalla (1977) and discussed in more detail by Ghosh and Ebrahimi (1980) lead to one or the other MHNWUE definition as introduced in section 2.

2. MHNWUE: definitions and example. Let  $X_1, \dots, X_p$  denote the survival (failure) times of  $p$  devices having a joint distribution function  $H_p(x_1, \dots, x_p)$ . The joint survival function of these  $p$  devices is denoted by  $\bar{H}_p(x_1, \dots, x_p) = P(X_1 > x_1, \dots, X_p > x_p)$ . It is assumed that  $\bar{H}_p(0, \dots, 0) = 1$ . In the univariate case, a non-negative random variable  $X_1$  is said to have a HNBUE (HNWUE) distribution if

$$\int_{t_1}^{\infty} \bar{H}_1(x) dx \leq (\geq) \mu \exp(-t_1/\mu) \text{ for all } t_1 \geq 0, \quad (2.1)$$

where it is assumed that  $\int_0^{\infty} \bar{H}_1(x) dx = \mu < \infty$ .

With the alternate representation of (2.1)

$$\frac{1}{\frac{1}{t_1} \int_0^{t_1} e_{H_1}(x) dx} \leq (\geq) \mu \text{ for all } t_1 \geq 0 \quad (2.2)$$

where  $e_{H_1}(x) = \frac{\int_x^\infty \bar{H}_1(y) dy}{\bar{H}_1(x)}$ , it is easy to see that the condition

is equivalent to saying that the integral harmonic mean value of the mean residual life of a unit at age  $x$  is less (greater) than or equal to the integral harmonic mean value of a new unit.

Definitions. Our definitions of MHNBU are the natural multivariate extension of (2.2). Suppose for simplicity that  $\bar{H}_p(x_1, \dots, x_p) > 0$  and let

$$e_{H_p}(x_1, \dots, x_p) = \frac{\int_{x_1}^\infty \dots \int_{x_p}^\infty \bar{H}_p(y_1, \dots, y_p) dy_1 \dots dy_p}{\bar{H}_p(x_1, \dots, x_p)}, \quad (2.3)$$

$$g_{H_p}(x, \dots, x) = \frac{\int_x^\infty \bar{H}_p(y, \dots, y) dy}{\bar{H}_p(x, \dots, x)}. \quad (2.4)$$

$H_p$  is said to be

(i) MHNBU-I (MHNWUE-I) if

$$\frac{\prod_{i=1}^p t_i}{\int_0^{t_1} \dots \int_0^{t_p} e_{H_p}^{-1}(x_1, \dots, x_p) dx_1, dx_2, \dots, dx_p} \leq (\geq) e_{H_p}(0, \dots, 0) \quad (2.5)$$

for all  $t_i \geq 0$  ( $1 \leq i \leq p$ ), and similar inequalities are assumed to hold for all subsets of random variables.

(ii) MHNBU-III (MHNWUE-III) if

$$\frac{t^p}{\int_0^t \dots \int_0^t e_{H_p}^{-1}(x_1, \dots, x_p) dx_1 \dots dx_p} \leq (\geq) e_{H_p}(0, \dots, 0) \quad (2.6)$$

for all  $t \geq 0$ , and similar inequalities hold for all subsets of random variables.

(iii) MHNBU-IV (MHNWUE-IV) if

$$\frac{t}{\int_0^t e_{H_p}^{-1}(x, \dots, x) dx} \leq (\geq) e_{H_p}(0, \dots, 0) \quad (2.7)$$

for all  $t \geq 0$ , and similar inequalities hold for all subsets of random variables.

(iv) MHNBU-V (MHNWUE-V) if

$$\frac{t}{\int_0^t g_{H_p}^{-1}(x, \dots, x) dx} \leq g_{H_p}(0, \dots, 0) \quad (2.8)$$

for all  $t \geq 0$ , and similar inequalities hold for all subsets of random variables.

Next we give the physical interpretation of these four definitions. First note that

$$e_{H_p}(x_1, \dots, x_p) = \int_{x_1}^{\infty} \dots \int_{x_p}^{\infty} \prod_{i=1}^p (y_i - x_i) dH(y_1, \dots, y_p), \quad (2.9)$$

for proof see Ghosh and Ebrahimi (1980). Now using (2.9) it follows

that (2.5) is equivalent to the statement that the integral harmonic mean value of the conditional mean residual product lifetime of the components of a unit with the components surviving ages  $x_1, \dots, x_p$  respectively is less (greater) than or equal to the integral harmonic mean value of the mean product lifetime of the components of a new unit. Similar interpretation holds for (2.6). The definition (2.7) is equivalent to the statement that the integral harmonic mean value of the conditional mean residual product lifetime of the components of a unit when all the components have survived a certain time  $x$  is less (greater) than or equal to the harmonic mean value of the mean product lifetime of the components of a new unit. Finally, definition (2.8) is equivalent to the statement that a multivariate distribution is HNBUE (HNWUE) if the minimum of the components has a univariate HNBUE distribution.

It is trivial to check that  $\text{MHNBU-E-I} \implies \text{MHNBU-E-II}$ . However, the following examples show that

$$\text{MHNBU-E-I} \not\Rightarrow \text{MHNBU-E-IV} \text{ (so that } \text{MHNBU-E-II} \not\Rightarrow \text{MHNBU-E-IV}).$$

Example 1. Let  $X_1$  and  $X_2$  be iid with common survival function

$$\bar{F}(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ 1/4 & \text{if } 3 \leq x < 7 \\ 0 & \text{if } x \geq 7 \end{cases}.$$

Then by using  $(p_2)$  of section 3,  $(X_1, X_2)$  is MHNBU-E-I. But  $\min(X_1, X_2)$  is not HNBUE, i.e.,  $(X_1, X_2)$  is not MHNBU-E-IV.

The next example shows that  $\text{MHNBU-E-I} \not\Rightarrow \text{MHNBU-E-III}$  (so that  $\text{MHNBU-E-II} \not\Rightarrow \text{MHNBU-E-IV}$ ).

Example 2. Let  $X_1$  and  $X_2$  be independent with survival functions

$$\bar{F}_{X_1}(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ 1/4 & \text{if } 3 \leq x < 7 \\ 0 & \text{if } x \geq 7 \end{cases} \quad \text{and} \quad \bar{F}_{X_2}(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 1/2 & \text{if } 1 \leq x < 5 \\ 0 & \text{if } x \geq 5 \end{cases}$$

respectively. By using  $(P_2)$  of section 3,  $(X_1, X_2)$  is MHNBE-I.

But  $(X_1, X_2)$  is not MHNBE-III.

Finally, the following example shows that

MHNBE-IV  $\not\Rightarrow$  MHNBE-III.

Example 3. Let  $X_1, X_2$  denote the survival times of 2 devices having a joint survival distribution function

$$\bar{H}_2(x_1, x_2) = \exp(-\max(\lambda_3 x_1, \lambda_4 x_2)),$$

where  $\lambda_3, \lambda_4 > 0$ . Then,  $(X_1, X_2)$  is MHNBE-IV. But  $(X_1, X_2)$  is not MHNBE-III.

Remark 1. Example 1 shows that MHNBE-III  $\not\Rightarrow$  MHNBE-IV.

Klefsjö's (1980) definitions of the MHNBE are the natural multivariate extension of (2.1).

We first introduce the following definitions of MHNBE as given in Klefsjö (1980). A bivariate distribution function  $H_2$  is said to be MHNBE if

$$A. \quad \int_0^\infty \int_0^\infty \bar{H}_2(x_1 + t_1, x_2 + t_2) dt_1 dt_2 \leq \int_0^\infty \int_0^\infty \bar{G}(x_1 + t_1, x_2 + t_2) dt_1 dt_2$$

for  $x_1, x_2 \geq 0$ ;

$$B. \quad \int_0^{\infty} \bar{H}_2(x_1 + t, x_2 + t) dt \leq \int_0^{\infty} \bar{G}(x_1 + t, x_2 + t) dt$$

for  $x_1, x_2 \geq 0$ ;

$$C. \quad \int_0^{\infty} \int_0^{\infty} \bar{H}_2(x + t_1, x + t_2) dt_1 dt_2 \leq \int_0^{\infty} \int_0^{\infty} \bar{G}(x + t_1, x + t_2) dt_1 dt_2$$

for  $x \geq 0$ ;

$$D. \quad \int_0^{\infty} \bar{H}_2(x + t, x + t) dt \leq \int_0^{\infty} \bar{G}(x + t, x + t) dt$$

for all  $x \geq 0$ .

In all cases  $\bar{G}(t_1, t_2)$  is the bivariate exponential distribution proposed by Marshall and Olkin (1967) where

$$\bar{G}(t_1, t_2) = \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} \max(t_1, t_2)) \text{ for } t_1, t_2 \geq 0,$$

and

$$\lambda_1 = \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_1}, \quad \lambda_2 = \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_2}, \quad \lambda_{12} = \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \frac{\mu_{12} - \mu_1 \mu_2}{\mu_{12}},$$

$$\mu_1 = \int_0^{\infty} \bar{H}_2(t_1, 0) dt_1, \quad \mu_2 = \int_0^{\infty} \bar{H}_2(0, t_2) dt_2, \quad \text{and} \quad \mu_{12} = \int_0^{\infty} \int_0^{\infty} \bar{H}(t_1, t_2) dt_1 dt_2.$$

Remark 2. The Klefsjö's definitions of MHNBE are restricted to the class of life distributions for which the following conditions hold among  $\mu_1$ ,  $\mu_2$ , and  $\mu_{12}$ :

$$(i) \quad \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_1} \geq 0, \quad (ii) \quad \frac{\mu_1 + \mu_2}{\mu_{12}} - \frac{1}{\mu_2} \geq 0, \quad \text{and}$$

$$(iii) \quad \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \left( \frac{\mu_{12} - \mu_1 \mu_2}{\mu_{12}} \right) > 0. \quad (2.10)$$

The following example shows that Klefsjö's definitions of MHNBU does not imply our definitions of MHNBU.

Example 4. Let  $X_1, X_2$  be iid and  $X_1, X_2$  be HNBUE. Choose  $\mu_1, \mu_2$ , and  $\mu_{12}$  which does not satisfy one of the conditions in (2.10). Then  $(X_1, X_2)$  satisfies our definitions, but it does not satisfy Klefsjö's definition of MHNBU.

3. MHNBU closure properties. In this section, we prove certain closure properties of MHNBU distribution. Let  $\beta_1, \beta_2, \beta_3$ , and  $\beta_4$  denote the classes of life distributions satisfying the definitions (i), (ii), (iii), and (iv) respectively of MHNBU. Then we have the following theorem.

Theorem 1.

- (P 1) If  $(T_1, \dots, T_m) \in \beta_j$ , any subset of  $(T_1, \dots, T_m) \in \beta_j$  ( $1 \leq j \leq 4$ );
- (P 2) If  $(T_1, \dots, T_m) \in \beta_j$ ,  $(T'_1, \dots, T'_n) \in \beta_j$  and are independent, then  $(T_1, \dots, T_m, T'_1, \dots, T'_n) \in \beta_j$  ( $1 \leq j \leq 2$ ). If  $(T_1, \dots, T_n) \in \beta_3$ ,  $(T'_1, \dots, T'_n) \in \beta_3$  are independent with identical distribution, then  $(T_1, \dots, T_n, T'_1, \dots, T'_n) \in \beta_3$ .
- (P 3) If  $(T_1, \dots, T_m) \in \beta_j$ , then  $(c_1 T_1, \dots, c_m T_m) \in \beta_j$  ( $1 \leq j \leq 2$ ) for all  $c_i > 0$  ( $1 \leq i \leq m$ ). If  $(T_1, \dots, T_m) \in \beta_j$ , then  $(c T_1, \dots, c T_m) \in \beta_j$  ( $2 \leq j \leq 4$ ) for all  $c > 0$ .

Proof.

(P 1): This property follows immediately from the definitions.

(P 2): Let  $\underline{a} = (a_1, \dots, a_m)$ , an  $m$  vector and  $\underline{a}' = (a'_1, \dots, a'_n)$ , an  $n$  vector. Other vectors can be defined similarly. Let

$$I(\underline{x}) = \int_0^{\underline{t}} [P(\underline{T} > \underline{x}) / \int_{\underline{x}}^{\infty} P(\underline{T} > \underline{y}) d\underline{y}] d\underline{x}, \text{ where } d\underline{x} = dx_1, \dots, dx_m \text{ and}$$

$\int_0^{\underline{t}}$  is a multiple integral with  $m$  factors.  $I(\underline{x}')$  and  $I(\underline{x}, \underline{x}')$  are

defined similarly. Then

$$\begin{aligned} & \prod_{i=1}^m t_i \prod_{j=1}^n t'_j / I(\underline{x}, \underline{x}') \\ & \left[ \prod_{i=1}^m t_i / I(\underline{x}) \right] \left[ \prod_{j=1}^n t'_j / I(\underline{x}') \right] \\ & \leq \int_0^{\infty} P(\underline{T} > \underline{x}) d\underline{x} \int_0^{\infty} P(\underline{T}' > \underline{x}') d\underline{x}' \\ & = \int_0^{\infty} \int_0^{\infty} P(\underline{T} > \underline{x}, \underline{T}' > \underline{x}') d\underline{x}' d\underline{x} \end{aligned}$$

Similar proof works for  $\beta_2$ . To prove  $\beta_3$ , use the Jensen inequality

(P 3): If  $\underline{T} \in \beta_1$ ,  $c_i > 0$  ( $1 \leq i \leq m$ )

$$\begin{aligned} & \prod_{i=1}^m t_i / \int_0^{\underline{t}} [P(\underline{d} > \underline{x}) / \int_0^{\infty} P(\underline{d} > \underline{y} + \underline{x}) d\underline{y}] d\underline{x} \\ & = \prod_{i=1}^m t_i / \int_0^{\underline{t}} [P(\underline{T} > \underline{e}) / \int_0^{\infty} P(\underline{T} > \underline{f}) d\underline{y}] d\underline{x} \\ & \leq \int_0^{\infty} P(\underline{T} > \underline{e}) d\underline{x} = \int_0^{\infty} P(\underline{d} > \underline{x}) d\underline{x}. \end{aligned}$$



Here  $\underline{d} = (c_1 T_1, \dots, c_m T_m)$ ,  $\underline{y} + \underline{x} = (y_1 + x_1, \dots, y_m + x_m)$ ,  
 $\underline{e} = (x_1/c_1, \dots, x_m/c_m)$  and  $\underline{f} = (y_1/c_1, \dots, y_m/c_m)$ . Similar proofs  
 work for  $\beta_2$ ,  $\beta_3$ , and  $\beta_4$ .

Remark 3. Since it is known in the univariate case that the HNBUE is not closed under the formation of coherent system (see Klefsjö (1980)), the same cannot be expected for the MHNBE under any of the four definitions.

Remark 4. Example 1 shows that (P 2) does not hold for  $\beta_4$ .

Remark 5. To prove that MHNBE is closed under limits in distribution, we need an extra condition to guarantee the application of the dominated convergence theorem.

Theorem 2. Let  $\{(T_{1k}, \dots, T_{mk}), k \geq 1\}$  be a sequence of MHNBE random vectors belonging to  $\beta_j$  for each  $k$ . If  $(T_{1k}, \dots, T_{mk}) \xrightarrow{\text{st}} (T_1, \dots, T_m)$  weakly as  $k \rightarrow \infty$  and  $(T_{1k}, \dots, T_{mk}) \leq (S_1, \dots, S_m)$  for all  $k \geq k_0$  where  $E(\sum_{i=1}^m S_i) < \infty$ , then  $(T_1, \dots, T_m) \in \beta_j$  for each  $j$ .

Proof. Use the dominated convergence theorem and the appropriate definition of the MHNBE.

It is known in the univariate case that HNBUE is closed under convolution. We have not been able to prove the same result for the multivariate case. Instead, we have proved the following theorem which is a special type of convolution.

Theorem 3. Let (a)  $\underline{X} = (X_1, \dots, X_n)$  and  $\underline{Y} = (Y_1, \dots, Y_n)$  be component-wise independent; (b)  $\underline{X}$  and  $\underline{Y}$  are independent; and (c)  $\underline{X}$  and  $\underline{Y} \in \beta_j$  for  $j = 1, 2$ . Then  $W = \underline{X} + \underline{Y} \in \beta_j$  for  $j = 1, 2$ .

Proof. Use (P 2) and the fact that  $X_i + Y_i$ 's are HNBUE.

Remark 5. In Theorem 3 if  $\tilde{X}$  and  $\tilde{Y}$  have the same distribution, then  $\tilde{W} = \tilde{X} + \tilde{Y} \in \beta_3$ .

#### 4. Relationship between MNBUE and MHNBU.

In the univariate case, a life distribution function  $H_1$  is said to satisfy the NBUE property if

$$\int_t^\infty \bar{H}(x) dx \leq \bar{H}_1(t) \int_0^\infty \bar{H}_1(x) dx \text{ for all } t \geq 0.$$

Rolski (1975), and Klefsjö (1980) have shown in the univariate case that NBUE  $\implies$  HNBUE, and that the converse implication does not hold. It is of interest to know whether a similar implication holds in the multidimensional case. Buchanan and Singpurwalla (1977) have given several definitions of the MNBUE based on multivariate generalization of the univariate NBUE distributions.

We first introduce the following definitions of MNBUE as given in Ghosh and Ebrahimi (1980). A  $p$ -variate distribution function  $H_p$  is said to be MNBUE if

A.

$$\int_{t_p}^\infty \int_{t_1}^\infty \bar{H}_p(x_1, \dots, x_p) dx_1 \dots dx_p \leq \bar{H}_p(t_1, \dots, t_p) \int_{t_p}^\infty \int_{t_1}^\infty \bar{H}_p(x_1, \dots, x_p) dx_1 \dots dx_p$$

for all  $t \geq 0$  ( $1 \leq i \leq p$ ), and similar inequalities are assumed to hold for all subsets.

B.

$$\int_t^\infty \dots \int_t^\infty \bar{H}_p(x_1, \dots, x_p) dx_1, \dots, dx_p \leq \bar{H}_p(t, \dots, t) \int_0^\infty \dots \int_0^\infty H_p(x_1, \dots, x_p) dx_1, \dots, dx_p$$

for all  $t \geq 0$ , and similar inequalities are assumed to hold for all subsets.

C.

$$\int_t^\infty \bar{H}_p(x, \dots, x) dx \leq \bar{H}_p(t, \dots, t) \int_0^\infty H_p(x, \dots, x) dx$$

for all  $t \geq 0$ , and similar inequalities hold for all subsets.

Ghosh and Ebrahimi (1980) have shown that the following implications hold:  $A \Rightarrow B$ ,  $A \not\Rightarrow C$ , and  $C \Rightarrow B$ .

We first explore the interrelationship of the A - C definitions of MNBUE with MHNBU-E-I to IV.

It is trivial to check that  $MHNBU-E-I \Rightarrow MHNBU-E-I \Rightarrow MNBUE-II$ ,  $MNBUE-II \Rightarrow MNBUE-III$ , and  $MNBUE-III \Rightarrow MNBUE-IV$ .

Example 5. ( $MNBUE-I \not\Rightarrow MNBUE-IV$ ). Consider once again example 1. Then  $(X_1, X_2)$  is MNBUE-I. But  $(X_1, X_2)$  is not MNBUE-IV.

Remark 6. The example 4 shows that  $MNBUE-II \not\Rightarrow MNBUE-IV$ .

Remark 7. Let  $X_1$  and  $X_2$  have bivariate Marshal-Olkin (1967) exponential distribution with survival function

$$\bar{H}_2(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)]$$

$x_1 \geq 0, x_2 \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_{12} > 0$ . Then  $\bar{H}_2(x_1, x_2)$  is both MNBUE-IV and MHNWUE-IV, MNBUE-III, and MHNWUE-III.

Remark 8. To see the interrelationship between MHN BUE and other classes of life distribution see Ghosh and Ebrahimi (1980). It is interesting to mention that multivariate IFRA defined by Block and Savits (1980) implies MHN BUE-IV.

The next example shows that MHN BUE class of life distributions is larger than the class of MNBUE life distributions.

Example 6. Let  $X_1, X_2$  are independent and identically distributed with the following survival distribution function,

$$\bar{F}(t) = \begin{cases} 1 & 0 \leq t < 1 \\ \frac{1}{2} & 1 \leq t < 2 \\ \frac{1}{8} & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$$

Then  $(X_1, X_2)$  is MHN BUE-I, MHN BUE-II, MHN BUE-III, and MHN BUE-IV.

But  $(X_1, X_2)$  is not MNBUE-I or MNBUE-II or MNBUE-III.

The following example shows that the MHNWUE class is also larger than the MNWUE class.

Example 7. Let  $X_1, X_2$  are iid with the following survival distribution,

$$\bar{F}(t) = \begin{cases} 1 & 0 \leq t < 1 \\ \frac{37}{192} & 1 \leq t < 2 \\ \frac{21}{192} & 2 \leq t < 3 \\ \frac{5}{192} & 3 \leq t < 4 \\ \left(\frac{1}{4}\right)^k & k \leq t < k+1 \text{ for } k = 4, \dots \end{cases}$$

Then  $(X_1, X_2)$  is MHNWUE-I, MHNWUE-II, MHNWUE-III, and MHNWUE-IV. But

$(X_1, X_2)$  is not MNWUE-I or MNWUE-II or MNWUE-III.

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